Application of a Generalized Frobenius-Perron Operator Scheme to Multivariate Nonlinear Dynamical Systems *

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The concept of generalized Frobenius-Perron operators is applied to multivariante nonlinear dynamical systems, and the associated generalized free energies are investigated. As important applications, diffusion-related free energies obtained from normally and superlinearly diffusive one-dimensional maps are discussed.

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The concept of describing dynamical systems with the help of free energies or associated entropies has proved to be very successful. Generally, dynamical systems are nonhyperbolic. This leads to the occurrence of the phenomenon of phase transitions. This phenomenon is easy to observe in the spectra of Lyapunov exponents, fractal dimensions and other entropy-like scaling functions [1–3]. In the associated free energies, the phase transitions are documented by the existence of critical lines. For maps on a grid of unit cells, nonhyperbolicity may also give rise to anomalous diffusion. The characteristic quantity to be investigated for diffusion [4–12] is the mean-squared displacement r as a function of the time t (or of the number of iterations n in the discrete case). In most cases it is found to behave as $\langle r^2(t)\rangle \sim t^{\alpha}$. For regular diffusion, which is characteristic for Brownian motion, $\alpha=1$. Then, the diffusive behavior is described by $\langle r^2(t)\rangle \cong D\,t$, where D is the diffusion coefficient. However, exponents different from one are frequently found in experiments. Then we speak of sublinear diffusion if $\alpha<1$, and of superlinear diffusion, if $\alpha>1$.

Since both phenomena are triggered by the nonhyperbolicity, it may be worthwhile working out the underlying principle for these effects on the basis of a multivariante thermodynamic formalism. A convenient was is to consider a generalized gradcanonical Frobenius-Perron operator [13] of the form

$$G_{\boldsymbol{\beta},z}\,\tilde{\Psi}(\boldsymbol{x}) := \sum_{n,\boldsymbol{y}:\,g_1(\boldsymbol{y})=x_1} \frac{z^{n(\boldsymbol{y})}\,\tilde{\Psi}(\boldsymbol{y})}{\prod_{i=1}^{N} |g_i'(\boldsymbol{y})|^{\beta_i}}\,, \tag{1}$$

where $g_i(\boldsymbol{y}) := f_i^{\,n}(y_i)$ and $n(\boldsymbol{y}) \le \bar{n} \to \infty$. In the formula, the map f_i with index $i=1$ denotes the dynamical map, and the prime indicates the derivation of g_i with respect to y_i. In this way, distinct measures of interest are included by chosing appropriate maps for $i>1$ $(1\le i\le N)$. The dynamical system is described macroscopically by a generalized free energy to which sometimes a manifest interpretation can be given (free energy of a spin system with interaction and $N-1$ external fields). This generalized free energy F_{G} can be obtained as

$$-F_{\mathrm{G}}(\boldsymbol{\beta}) = \log_e(z_m), \tag{2}$$

where z_m is the smallest positive root of the equation

$$\sum_{n,\boldsymbol{y}:\,g_1(\boldsymbol{y})=x_1} \frac{z^{n(\boldsymbol{y})}}{\prod_{i=1}^{N} |g_i'(\boldsymbol{y})|^{\beta_i}} = 1\,, \tag{3}$$

(for expanding systems, the dependence upon x vanishes for $\bar{n}\to\infty$). Points of nonanalytical behavior of the free energy emergy generically when the order of the leading two eigenvalues of (1) changes upon variation of some parameter β_i (the temperature or the external fields). Since both eigenvalues are smooth functions of these parameters, a codim-1 critical surface will evolve. Generically, the phase transition at

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this surface will be of first order. However, if the nonhyperbolicity can be reached by a sufficiently smooth limiting process of the hyperbolic elements, as it is the case for intermittent systems, this transition may also be of higher order. In numerical simulations it is often difficult to distinguish first from higher oder phase transitions. In some cases, however, drastic consequences of phase transitions can be observed. For the free energy of some simple models, a lower bound of the following form has been proved [2]:

$$F_G(\boldsymbol{\beta}) \geq \sum_{i=1}^{N} \beta_i \frac{\log_e(c_i)}{z_i}, \tag{4}$$

where z_i is the order of the maximum of the smooth one-humped map f_i, and c_i are the slopes of its left hand fixed point. If one of the slopes c_i is equal to unity or infinity we obtain a so-called exotic phase transition. In this case, the free energy may be confined to a certain range of values, in contrast to the hyperbolic case, and we may even have a jump of infinite height if one parameter β_i is varied. As a result, the behavior of the free energy is determined by the dynamics of the system, by the choices of the measure map and their analytically, and symmetry properties. The influence of nonhyperbolicity is indicated in the free energy by critical lines and by possible infinite jumps. Due to the arbitrariness of the choice of the measure map, many distinct properties of a given system may be investigated, isolated from the other properties of interest or in combination with them. The general procedure yields a vast field of applications which is far from having been exploited. In the following, in order to demonstrate this fact, we will discuss to some extent two applications (for these examples, $N=2$).

As a first application we consider the free energy obtained from bivariate models used by one of the authors to discuss the relations between phase transition behavior in different entropy-like scaling functions ([3], see also [2]). In this case, in addition to the dynamical map f_1, which is interpreted as the map of the support, a measure map f_2 is used to describe the probability p with which a certain symbol s of the associated symbolic description is assumed:

$$(|g_2'(y)|^{\beta_2})^{-1} = p(s(y))^q \tag{5}$$

(traditionally, β_2 is denoted by q). In one-dimensional hyperbolic cases, this measure can be obtained from the usual monovariate free energy of length scales (Ruelle's topological pressure, $\beta_i=0$, $i>1$). However,

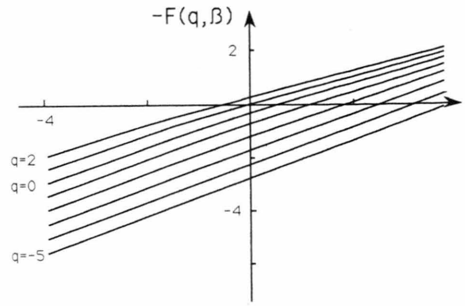

Fig. 1. Free energy of a hyperbolic system generated by two asymmetric tent maps (absolute slopes $\frac{9}{2}$ and $\frac{7}{2}$, probabilities $\frac{3}{10}$ and $\frac{7}{10}$. The probabilities are obtained as the inverses of the absolute slopes of the second tent map). There are no forbidden zones.

in order to model more general situations, the generalization indicated above (known as a Cantor set with measure) is useful [2–3]. If both maps are hyperbolic (e.g., both are tent maps with absolute slopes larger than unity), the system is called hyperbolic. For hyperbolic systems, the free energy is analytic. In Fig. 1 an example of such a behavior is shown. For nonhyperbolic systems, a critical line emerges in the free energy [2, 3b–c). The free energy, or in a more condensed form the associated entropy, contains all information on the probabilistic or dynamical invariants like Renyi entropies, fractal dimensions, and Lyapunov exponents [3d–e]. Using a different measure map, with the same concept a system with diffusion can be described [14, 15]. For the diffusion, the ability of inducing a number of consecutive jumps is distributed on the elements of the partition of the interval in a measurelike fashion. However, since two directions of motion are possible, we rather deal with a signed measure instead of the usual concept. If the map on a unit cell has a point-symmetry in order to avoid a possible drift, this fact is reflected in the properties of the free energy, as we will see below. In the diffusion-related application, let us consider systems for which the probabilistic information is included in Ruelle's pressure, so that no additional measure is needed. More general situations may be easily thought of. With respect to the general exposition, the earlier bivariate and the diffusion-related cases differ from distinct choices of the map g_2 (different potentials in addition to the interaction energy determined by g_1). In order to calculate the diffusion-related free energy, the partition sum (3) has to be specified. To each element of the partition, a measure of the ability to induce jumps to

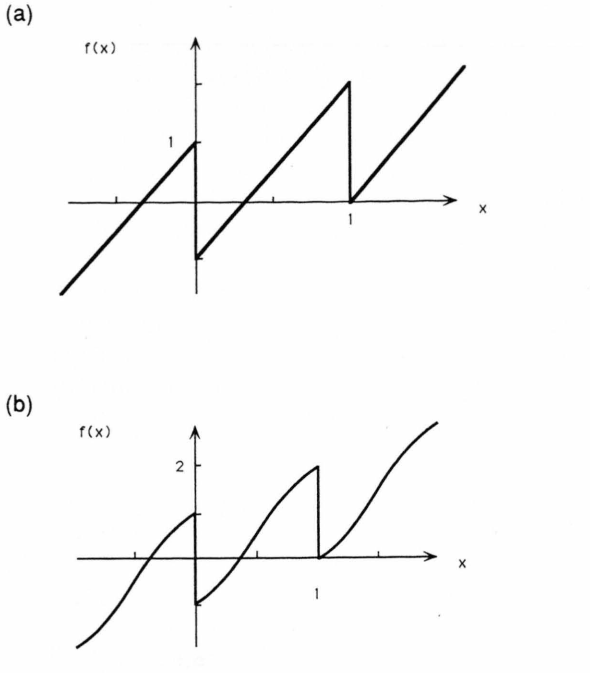

Fig. 2. a) Roof map, b) intermittent map which leads to superlinear diffusion. The roof map is obtained from the intermittent map in the limit $\bar{z}\to 1$, where $\bar{z}$ is the intermittency exponent. In b), $\bar{z}=5/3$.

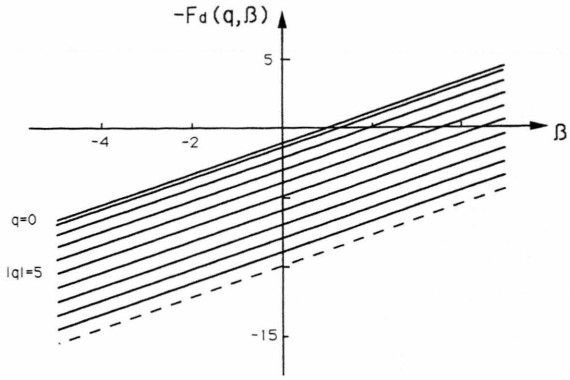

Fig. 3. Diffusion-related free energy of the (hyperbolic) roof map.

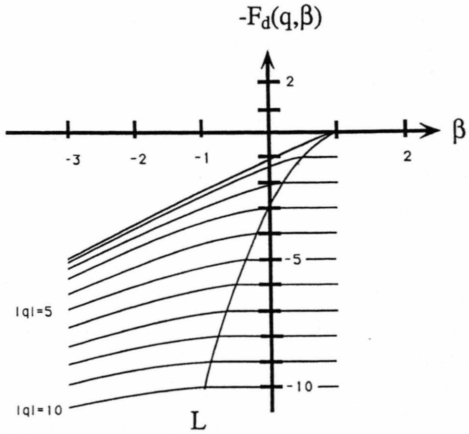

Fig. 4. Bivariate diffusion-related free energy for the superlinear case ($\bar{z}=2$). Note the existence of the critical line L and a second forbidden zone below the β-axis beyond $\beta=1$ (see [13]).

neighboring boxes within n iterations should be assigned. This is achieved by use of the weight

$$|g_2'(y)|^{\beta_2} = e^{-q\mu(I)g_1(y)))}, \tag{6}$$

where the function $\mu(I g_1))$ measures the number of visits of the elements which are responsible for the transport. Transport events to right-hand side neighboring cells are counted with positive sign, those to the left-hand side are counted with negative sign (compare, e.g., [12]). In principle, for the numerical calculation of the free energy the canonical or the grand-canonical partition sum can be taken. However, for many applications (e.g., intermittency or for systems with nontrivial grammars), the use of the grandcanonical transfer operator is advantageous [13–14].

In the free energy of a completely hyperbolic diffusive system, no nonanalytic behavior is possible. For the two maps in Fig. 2, the diffusion-related generalized free energies will be discussed. Whereas for the hyperbolic map shown in Fig. 2a a normally diffusive behavior is obtained, the nonhyperbolic map in Fig. 2b leads to superlinear diffusion. In Fig. 3, the diffusion-related free energy of the roof map is displayed. In this figure, the above-mentioned analyticity and the symmetry properties are clearly visible. Furthermore, above the curve obtained for $q=0$ we can observe a forbidden zone which is due to the symmetry in q. The diffusion-related free energy of the diffusive symmetric tend map looks qualitatively the same, with generally different slopes of the straight lines, due to distinct topological entropies. (If the roof map of Fig. 2b is compared with the example of [12], the topological entropies are $\log_e$ (3) and $\log_e$ (7), respectively.) For a nonhyperbolic map the occurrence of a critical line is expected [2, 3b]. Figure 4 shows the free energy obtained for an intermittent, superlinearly diffusive map. As expected, a critical line (indicated in the figure by L) separates the hyperbolic aspect of the free energy on its left hand side from the nonhyperbolic aspect on the right hand side. Furthermore, parallel straight lines characterize the nonhyperbolic aspect. Only minimal differences in the free energies are observable if different examples of superlinearly diffusive maps with the same intermittency exponent $\bar{z}$ are compared. This topic is discussed in [13], where a detailed description of the used numerical procedures in given. In the following let us sketch how in the free energy all the relevant information of the system is contained. The fact that the point P with coordinates (1, 0) and a part of the negative vertical axis are included in the plots of the free energy is not accidental. The topological entropy of the system may be obtained as the absolute value of the intersection point of $F_G(q=0, \beta)$ with the y-axis. The Lyapunov exponent can be calculated as the left hand side derivative at the point P, so that this point must be included in the plot if the Lyapunov exponent of the system is required to exist. Furthermore, the diffusion coefficient D is determined by the second left hand side derivative at point P [16–17]

$$D = \frac{1}{2} \frac{\partial^2}{\partial q^2} F_d(q, \beta)|_{q=0, \beta=1} \tag{7}$$

(note that $\beta = 1$ selects the natural measure). Since for the existence/nonexistence of the diffusion coefficient the "weight" of the nonhyperbolic aspect in comparison to the hyperbolic aspect is relevant, the critical line has to pass through the point P. More generally, since the diffusion coefficient depends on the diffusion-related free energy for $q \neq 0$, the sublinear and the superlinear cases exhibit distinct diffusional properties, although the two processes may be described by the same reduced map [13, 14]. The behavior of the free energy at the critical lines can be worked out with the following result: The order of the phase transition which emerges there depends on the size of the intermittency exponent $\bar{z}$ of the system and on the values of β and q at the transition points (the transition is of increasing order for increasing $\bar{z}$ and of decreasing order in β and q). The relation with the diffusion coefficient D is for both cases of anomalous diffusion the following: If the phase transition is of second or higher order, then anomalous diffusion is observed. Otherwise, normal diffusion is obtained.

We conclude with a summarizing remark on the value of the intermittency exponent $\bar{z}$ for which anomalous diffusional behavior first appears. This value is different for the sublinear and the superlinear case. The diffusion coefficient can be calculated according to (7). For the generic sublinear intermittent system it is obtained that

$$D \sim \left(\sum_{k=1}^{\infty} k w_k \right)^{-1}, \tag{8}$$

where $w_k \sim k^{-(1/(\bar{z}-1)+1)}$. Obviously, the sum is finite for $\bar{z} < 2$ (note that the sum actually measures the average escape time [18]). For the superlinear case, the evaluation of the diffusion coefficient yields an expression with the dominant summand

$$D \sim \frac{\sum_{k=1}^{\infty} k^2 w_k}{\sum_{k=1}^{\infty} k w_k}. \tag{9}$$

This explains why, for the superlinear case, anomality sets in at $\bar{z} = 3/2$. From both expression for D, due to the properties of w_k, the correct behavior of the mean squared displacement can be recovered.

A more detailed study of diffusion with emphasis on the numerical procedures and on the derivation of the diffusional properties will be published elsewhere [13]. In future, it is planned to apply the formalism to diffusion in Hamiltonian systems. We believe that similar concepts can be useful for other measures of interest.

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[1] P. Szépfalusy and T. Tél, Phys. Rev. **A 34,** 2520 (1986).
[2] Z. Kovács and T. Tél, Phys. Rev. **A 45,** 2270 (1992).
[3] a) R. Stoop and J. Parisi, Phys. Rev. **A 43,** 1802 (1991), b) R. Stoop, Z. Naturforsch. **46 a,** 1117 (1991), c) R. Stoop, Phys. Rev. **A 46,** 7450 (1992), d) R. Stoop, Phys. Lett. **A 173,** 369 (1993), e) R. Stoop, Phys. Rev. **E 47,** 3927 (1993).
[4] H. Scher and E. W. Montroll, Phys, Rev. **B 12,** 2455 (1975).
[5] M. F. Shlesinger, J. Klafter, and Y. M. Wong, J. Stat. Phys. **27,** 499 (1982).
[6] G. Zumofen, J. Klafter, and A. Blumen, J. Stat. Phys. **65,** 991 (1991).
[7] M. Araujo, S. Havlin, G. H. Weiss, and H. E. Stanley, Phys. Rev. **A 43,** 5207 (1991).
[8] T. Geisel and S. Thomae, Phys. Rev. Lett. **52,** 1936 (1984); T. Geisel, J. Nierwetberg and A. Zacherl, Phys. Rev. Lett. **54,** 616 (1985).
[9] J. Klafter, A. Blumen, and M. F. Shlesinger, Phys. Rev. **A 35,** 3081 (1987).
[10] T. Geisel, J. Wagenhuber, P. Niebauer, and G. Obermair, Phys. Rev. **B 45,** 4372 (1992).
[11] J. Wagenhuber, T. Geisel, P. Niebauer, and G. Obermair, Phys. Rev. Lett. **64,** 1581 (1990).
[12] X.-J. Wang, Phys. Rev. **A 45,** 8407 (1992).
[13] R. Stoop, to be published in Phys. Rev. E.
[14] X.-J. Wang and C.-K. Hu, Phys. Rev. **E48,** 728 (1993).
[15] R. Stoop, Europhys, Lett. **25,** 99 (1994).
[16] R. Artuso, Physics Lett. **A 160,** 530 (1991).
[17] P. Cvitanović, P. Gaspard, and T. Schreiber, Chaos **2,** 85 (1992).
[18] R. Stoop, G. Zumofen, and J. Parisi, Chaos, solitons, and fractals, to be published.